

Long time existence of regular solutions to 3d Navier-Stokes equations coupled with the heat convection

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Abstract. We prove long time existence of regular solutions to the Navier-Stokes equations coupled with the heat equation. We consider the system in non-axially symmetric cylinder with the slip boundary conditions for the Navier-Stokes equations and the Neumann condition for the heat equation. The long time existence is possible because we assumed that derivatives with respect to the variable along the axis of the cylinder of the initial velocity, initial temperature and the external force in L_2 norms are sufficiently small. We proved the existence of such solutions that velocity and temperature belong to $W_{\sigma}^{2,1}(\Omega \times (0, T))$, where $\sigma > \frac{5}{3}$. The existence is proved by the Leray-Schauder fixed point theorem.

Key words: Navier-Stokes equations, heat equation, coupled, slip boundary conditions, the Neumann condition, long time existence, regular solutions

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1. Introduction

We consider the following problem

$$\begin{aligned}
 (1.1) \quad & v_{,t} + v \cdot \nabla v - \operatorname{div} \mathbf{T}(v, p) = \alpha(\theta)f \quad \text{in } \Omega^T = \Omega \times (0, T), \\
 & \operatorname{div} v = 0 \quad \text{in } \Omega^T, \\
 & \theta_{,t} + v \cdot \nabla \theta - \varkappa \Delta \theta = 0 \quad \text{in } \Omega^T, \\
 & \bar{n} \cdot \mathbf{D}(v) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 \quad \text{on } S^T = S \times (0, T), \\
 & \bar{n} \cdot \bar{v} = 0 \quad \text{on } S^T, \\
 & \bar{n} \cdot \nabla \theta = 0 \quad \text{on } S^T, \\
 & v|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0 \quad \text{in } \Omega,
 \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$ is cylindrical domain, $S = \partial\Omega$, $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$ is the velocity of the fluid motion, $p = p(x, t) \in \mathbb{R}^1$ the pressure, $\theta = \theta(x, t) \in \mathbb{R}_+$ the temperature, $f = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$ the external force field, \bar{n} is the unit outward normal vector to the boundary S , $\bar{\tau}_\alpha$, $\alpha = 1, 2$, are tangent vectors to S and the dot denotes the scalar product in \mathbb{R}^3 . We define the stress tensor by

$$\mathbf{T}(v, p) = \nu \mathbf{D}(v) - p \mathbf{I},$$

where ν is the constant viscosity coefficient, \mathbf{I} is the unit matrix and $\mathbf{D}(v)$ is the dilatation tensor of the form

$$\mathbf{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

Finally \varkappa is a positive heat conductivity coefficient.

By $x = (x_1, x_2, x_3)$ we denote the Cartesian coordinates, $\Omega \subset \mathbb{R}^3$ is a cylindrical type domain parallel to the axis x_3 with arbitrary cross section.

We assume that $S = S_1 \cup S_2$, where S_1 is the part of the boundary which is parallel to the axis x_3 and S_2 is perpendicular to x_3 . Hence

$$S_1 = \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) = c_*, -b < x_3 < b\}$$

and

$$S_2 = \{x \in \mathbb{R}^3 : \varphi_0(x_1, x_2) < c_*, x_3 \text{ is equal either to } -b \text{ or } b\},$$

where b, c_* are positive given numbers and $\varphi_0(x_1, x_2)$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$. We can assume $\bar{\tau}_1 = (\tau_{11}, \tau_{12}, 0)$, $\bar{\tau}_2 = (0, 0, 1)$ and $\bar{n} = (\tau_{12}, -\tau_{11}, 0)$ on S_1 .

Assume $\alpha \in C^2(\mathbb{R}_+)$ and Ω^T satisfies the weak l -horn condition, where $l = (2, 2, 2, 1)$.

Moreover assume Ω is not axially symmetric. Now we formulate the main result of this paper. Let $g = f_{,x_3}$, $j = v_{,x_3}$, $q = p_{,x_3}$, $\vartheta = \theta_{,x_3}$, $\chi = (\operatorname{rot} v)_3$, $F = (\operatorname{rot} f)_3$. Assume that $\|\theta(0)\|_{L_\infty(\Omega)} < \infty$.

Define

$$a : [0, \infty) \rightarrow [0, \infty), \quad a(x) = \sup\{|\alpha(y)| + |\alpha'(y)| + |\alpha''(y)| : |y| \leq x\}$$

and $c_1 = a(\|\theta(0)\|_{L_\infty})$. Moreover assume that $\frac{5}{3} < \sigma < \infty$, $\frac{5}{3} < \varrho < \infty$, $\frac{5}{\varrho} - \frac{5}{\sigma} < 1$ and for $t \leq T$

1. $c_1\|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1c_0\|f\|_{L_\infty(0,t;L_3(\Omega))} + c_1\|F\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)} + \|\chi(0)\|_{L_2(\Omega)} + c_0^2(c_1\|f\|_{L_2(0,t;L_{6/5}(\Omega))} + \|v(0)\|_{L_2(\Omega)}) \leq k_1 < \infty$,
2. $\|f\|_{L_2(0,t;L_3(\Omega))} \leq k_2 < \infty$,
3. $\|f\|_{L_2(\Omega^t)} + \|v_0\|_{H^1(\Omega)} \leq k_3 < \infty$,
4. $c_1\|f\|_{L_\infty(\Omega^t)}e^{cc_1^2k_2^2}k_1 + c_1\|g\|_{L_\sigma(\Omega^t)} + \|\vartheta(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)} \leq k_4 < \infty$,
5. $c_1\|g\|_{L_2(0,t;L_{6/5}(\Omega))} + c_1\|f_3\|_{L_2(0,t;L_{4/3}(S_2))} + \|h(0)\|_{L_2(\Omega)} + \|\vartheta(0)\|_{L_2(\Omega)} \leq d < \infty$,
6. $c_1\|f\|_{L_\sigma(\Omega^T)} + \|v(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} + \|\theta(0)\|_{W_\varrho^{2-2/\varrho}(\Omega)} \leq k_5 < \infty$,

where c_0 is a constant from Lemma 2.2. Assume

$$f \in L_\sigma(\Omega^T), \quad g \in L_\sigma(\Omega^T),$$

$$\vartheta(0) \in W_\sigma^{2-2/\sigma}(\Omega).$$

Theorem 1.1. *Let the above assumptions hold. Assume that d is sufficiently small (see [5, Main Theorem]). Then there exists a strong solution (v, p, θ) to (1.1) such that $v, \theta \in W_\varrho^{2,1}(\Omega^T)$, $\nabla p \in L_\varphi(\Omega^T)$, $h, \vartheta \in W_\sigma^{2,1}(\Omega^T)$, $\nabla q \in L_\sigma(\Omega^T)$.*

The result follows by applying the methods developed in [3] to the more complicated system (1.1). However, the proof of existence in this paper is much more clear than the one in [3], because the mapping ϕ in this paper is constructed in a simpler way than the corresponding mapping in [3]. This, however, needs more regularity. Therefore in this paper we proved the existence of much more regular solutions than in [3].

2. Preliminaries

In this section we introduce notation and basic estimates for weak solutions to problem (1.1).

2.1. Notation

We use isotropic and anisotropic Lebesgue spaces: $L_p(Q)$, $Q \in \{\Omega^T, S^T, \Omega, S\}$, $p \in [1, \infty]$; $L_q(0, T; L_p(Q))$ $Q \in \{\Omega, S\}$, $p, q \in [1, \infty]$; Sobolev spaces

$$W_q^{s, s/2}(Q^T), \quad Q \in \{\Omega, S\}, \quad q \in [1, \infty], \quad s \in \mathbb{N} \cup \{0\}$$

with the norm

$$\|u\|_{W_q^{s, s/2}(Q^T)} = \left(\sum_{|\alpha|+2a \leq s} \int_{Q^T} |D_x^\alpha \partial_t^a u|^q dx dt \right)^{1/q},$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $a, \alpha_i \in \mathbb{N} \cup \{0\}$.

In the case $q = 2$

$$H^s(Q) = W_2^s(Q), \quad H^{s, s/2}(Q^T) = W_2^{s, s/2}(Q^T), \quad Q \in \{\Omega, S\}.$$

Moreover, $L_2(Q) = H^0(Q)$, $L_p(Q) = W_p^0(Q)$, $L_p(Q^T) = W_p^{0,0}(Q^T)$.

We define a space natural for study weak solutions to the Navier-Stokes and parabolic equations

$$V_2^k(\Omega^T) = \left\{ u : \|u\|_{V_2^k(\Omega^T)} = \operatorname{esssup}_{t \in [0, T]} \|u\|_{H^k(\Omega)} + \left(\int_0^T \|\nabla u\|_{H^k(\Omega)}^2 dt \right)^{1/2} < \infty \right\}.$$

2.2. Weak solutions

By a weak solution to problem (1.1) we mean $v \in V_2^0(\Omega^T)$, $\theta \in V_2^0(\Omega^T) \cap L_\infty(\Omega^T)$ satisfying the integral identities

$$\begin{aligned} (2.1) \quad & - \int_{\Omega^T} v \cdot \varphi_{,t} dx dt + \int_{\Omega^T} v \cdot \nabla v \cdot \varphi dx dt + \frac{\nu}{2} \int_{\Omega^T} \mathbf{D}(v) \cdot \mathbf{D}(\varphi) dx dt \\ & = \int_{\Omega^T} \alpha(\theta) f \cdot \varphi dx dt + \int_{\Omega} v_0 \varphi(0) dx, \end{aligned}$$

$$\begin{aligned} (2.2) \quad & - \int_{\Omega^T} \theta \psi_{,t} dx dt + \int_{\Omega^T} v \cdot \nabla \theta \psi dx dt + \varkappa \int_{\Omega^T} \nabla \theta \cdot \nabla \psi dx dt \\ & = \int_{\Omega} \theta_0 \psi(0) dx, \end{aligned}$$

which hold for $\varphi, \psi \in W_2^{1,1}(\Omega^T) \cap L_5(\Omega^T)$ such that $\varphi(T) = 0$, $\psi(T) = 0$, $\operatorname{div} \varphi = 0$, $\varphi \cdot \bar{n}|_S = 0$.

Lemma 2.1. (see [9]) (the Korn inequality) Assume that

$$(2.3) \quad E_{\Omega}(v) = \|\mathbf{D}(v)\|_{L_2(\Omega)}^2 < \infty, \quad v \cdot \bar{n}|_S = 0, \quad \operatorname{div} v = 0.$$

If Ω is not axially symmetric there exists a constant c_1 such that

$$(2.4) \quad \|v\|_{H^1(\Omega)}^2 \leq c_1 E_{\Omega}(v).$$

If Ω is axially symmetric, $\eta = (-x_2, x_1, 0)$, $\alpha = \int_{\Omega} v \cdot \eta dx$, then there exists a constant c_2 such that

$$(2.5) \quad \|v\|_{H^1(\Omega)}^2 \leq c_2 (E_{\Omega}(v) + |\alpha|^2).$$

Let us consider the problem

$$(2.6) \quad \begin{aligned} h_{,t} - \operatorname{div} \mathbf{T}(h, q) &= f && \text{in } \Omega^T, \\ \operatorname{div} h &= 0 && \text{in } \Omega^T, \\ \bar{n} \cdot h = 0, \quad \bar{n} \cdot \mathbf{D}(h) \cdot \bar{\tau}_{\alpha} &= 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\ h_i = 0, \quad i = 1, 2, \quad h_{3,x_3} &= 0 && \text{on } S_2^T, \\ h|_{t=0} &= h(0) && \text{in } \Omega \end{aligned}$$

Theorem 2.1. Let $f \in L_p(\Omega^T)$, $h(0) \in W_p^{2-2/p}(\Omega)$, $s \in C^2$, $1 < p < \infty$. Then there exists a solution to problem (2.6) such that $h \in W_p^{2,1}(\Omega^T)$, $\nabla q \in L_p(\Omega^T)$ and there exists a constant c depending on S and p such that

$$(2.7) \quad \|h\|_{W_p^{2,1}(\Omega^T)} + \|\nabla q\|_{L_p(\Omega^T)} \leq c(\|f\|_{L_p(\Omega^T)} + \|h(0)\|_{W_p^{2-2/p}(\Omega)}).$$

The proof follows from considerations from [2, Ch. 4].

Let us consider the problem

$$(2.8) \quad \begin{aligned} v_{,t} - \operatorname{div} \mathbf{T}(v, q) &= f && \text{in } \Omega^T, \\ \operatorname{div} v &= 0 && \text{in } \Omega^T, \\ \bar{n} \cdot v = 0, \quad \bar{n} \cdot \mathbf{D}(v) \cdot \bar{\tau}_{\alpha} &= 0, \quad \alpha = 1, 2, && \text{on } S^T, \\ v|_{t=0} &= v_0 && \text{in } \Omega. \end{aligned}$$

Theorem 2.2. (the proof is similar to the proof from [1]) Let $f \in L_p(\Omega^T)$, $v(0) \in W_p^{2-2/p}(\Omega)$, $S \in C^2$, $1 < p < \infty$. Then there exists a solution to problem (2.8) such that $v \in W_p^{2,1}(\Omega^T)$, $\nabla p \in L_p(\Omega^T)$ and there exists a constant c depending on S and p such that

$$(2.9) \quad \|v\|_{W_p^{2,1}(\Omega^T)} + \|\nabla p\|_{L_p(\Omega^T)} \leq c(\|f\|_{L_p(\Omega^T)} + \|v(0)\|_{W_p^{2-2/p}(\Omega)}).$$

Lemma 2.3. (see [5, Lemma 2.3])

Assume that $v_0 \in L_2(\Omega)$, $\theta_0 \in L_\infty(\Omega)$, $f \in L_2(0, T; L_{6/5}(\Omega))$, $T < \infty$. Assume that Ω is not axially symmetric. Assume that there exists constants θ_* , θ^* such that $\theta_* < \theta^*$ and

$$\theta_* \leq \theta_0(x) \leq \theta^*, \quad x \in \Omega.$$

Then there exists a weak solution to problem (1.1) such that $(v, \theta) \in V_2^0(\Omega^T) \times V_2^0(\Omega^T)$, $\theta \in L_\infty(\Omega^T)$ and

$$(2.10) \quad \theta_* \leq \theta(x, t) \leq \theta^*, \quad (x, t) \in \Omega^T,$$

$$(2.11) \quad \|v\|_{V_2^0(\Omega^T)} \leq c(a(\|\theta_0\|_{L_\infty(\Omega)})\|f\|_{L_2(0, T; L_{6/5}(\Omega))} + \|v_0\|_{L_2(\Omega)}) \leq c_0,$$

$$(2.12) \quad \|\theta\|_{V_2^0(\Omega^T)} \leq c\|\theta_0\|_{L_2(\Omega)} \leq c_0.$$

Remark 2.4. If $\theta(0) \geq 0$, then $\theta(t) \geq 0$ for $t \geq 0$.

3. Existence

For $\xi, \eta \geq 1$, $\sigma, \varrho \geq 1$ define

$$\begin{aligned} \|(v, \theta)\|_{\mathcal{M}(\Omega^T)} &= \|v\|_{L_\infty(0, T; W_\eta^1(\Omega))} + \|\theta\|_{L_\infty(0, T; W_\eta^1(\Omega))} \\ &\quad + \|v_{,x_3}\|_{L_\infty(0, T; W_\xi^1(\Omega))} + \|\theta_{,x_3}\|_{L_\infty(0, T; W_\xi^1(\Omega))}, \end{aligned}$$

$$\mathcal{M}(\Omega^T) = \{(v, \theta) : \|(v, \theta)\|_{\mathcal{M}(\Omega^T)} < \infty\},$$

$$\begin{aligned} \|(v, \theta)\|_{\mathcal{N}(\Omega^T)} &= \|v\|_{W_\varrho^{2,1}(\Omega^T)} + \|\theta\|_{W_\varrho^{2,1}(\Omega^T)} \\ &\quad + \|v_{,x_3}\|_{W_\sigma^{2,1}(\Omega^T)} + \|\theta_{,x_3}\|_{W_\sigma^{2,1}(\Omega^T)}, \end{aligned}$$

$$\mathcal{N}(\Omega^T) = \{(v, \theta) : \|(v, \theta)\|_{\mathcal{N}(\Omega^T)} < \infty\}.$$

Lemma 3.1. We have

1. $(\mathcal{M}(\Omega^T), \|\cdot\|_{\mathcal{M}(\Omega^T)})$ is the Banach space.
2. $(\mathcal{N}(\Omega^T), \|\cdot\|_{\mathcal{N}(\Omega^T)})$ is the Banach space.
3. $\|u\|_{\mathcal{M}(\Omega^T)} \leq c\|u\|_{\mathcal{N}(\Omega^T)}$ for $u \in \mathcal{N}(\Omega^T)$ and the imbedding $\mathcal{N}(\Omega^T) \subset \mathcal{M}(\Omega^T)$ is compact for $\varrho < \eta$, $\frac{5}{\varrho} - \frac{3}{\eta} < 1$, $\sigma < \xi$, $\frac{5}{\sigma} - \frac{3}{\xi} < 1$.

Let us consider the problems

$$\begin{aligned} (3.1) \quad & v_t - \operatorname{div} \mathbf{T}(v, p) = -\lambda[\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta})f], \\ & \operatorname{div} v = 0, \\ & v \cdot \bar{n}|_S = 0, \quad \bar{n} \cdot \mathbf{D}(v) \cdot \bar{\tau}_\alpha|_S = 0, \quad \alpha = 1, 2, \\ & v|_{t=0} = v_0 \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} \theta_t - \varkappa \Delta \theta &= -\lambda \tilde{v} \cdot \nabla \tilde{\theta}, \\ \bar{n} \cdot \nabla \theta|_S &= 0, \\ \theta|_{t=0} &= \theta_0, \end{aligned}$$

where $\lambda \in [0, 1]$ is parameter and $\tilde{v}, \tilde{\theta}$ are treated as given functions. We will assume that $\alpha \in C^2(\mathbb{R})$.

Lemma 3.2. *Assume that*

$$\begin{aligned} (\tilde{v}, \tilde{\theta}) &\in \mathcal{M}(\Omega^T), \quad 3 < \eta < \infty, \\ f &\in L_\varrho(\Omega^T), \quad 1 < \varrho < \infty \\ v_0 &\in W_\varrho^{2-2/\varrho}(\Omega), \\ S &\in C^2, \quad \frac{5}{\varrho} - \frac{3}{\eta} < 1, \quad \varrho < \eta. \end{aligned}$$

Then there exists a unique solution to problem (3.1) such that

$$v \in W_\varrho^{2,1}(\Omega^T) \subset L_\infty(0, T; W_\eta^1(\Omega))$$

and

$$\begin{aligned} \|v\|_{L_\infty(0, T; W_\eta^1(\Omega))} &\leq c \|v\|_{W_\varrho^{2,1}(\Omega^T)} \leq c(\lambda \|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2 \\ &\quad + \lambda a(c \|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}) \|f\|_{L_\varrho(\Omega^T)} + \|v_0\|_{W_\varrho^{2-2/\varrho}(\Omega)}). \end{aligned}$$

Proof. We have

$$\begin{aligned} \|\tilde{v} \cdot \nabla \tilde{v}\|_{L_\varrho(\Omega^T)} &\leq c \|\tilde{v}\|_{L_\infty(\Omega^T)} \|\nabla \tilde{v}\|_{L_\eta(\Omega^T)} \\ &\leq c \|\tilde{v}\|_{L_\infty(0, T; W_\eta^1(\Omega))}^2 \leq c \|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2. \end{aligned}$$

and

$$\begin{aligned} \|\alpha(\tilde{\theta})f\|_{L_\varrho(\Omega^T)} &\leq a(c \|\tilde{\theta}\|_{L_\infty(0, T; W_\eta^1(\Omega))}) \|f\|_{L_\varrho(\Omega^T)} \\ &\leq c(c \|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}) \|f\|_{L_\varrho(\Omega^T)}. \end{aligned}$$

By Theorem 2.2 the proof is completed. □

Lemma 3.3. *Assume that*

$$\begin{aligned} 3 < \eta < \infty, \quad 1 < \varrho < \infty, \quad \varrho < \eta, \quad \frac{5}{\varrho} - \frac{3}{\eta} < 1, \\ (\tilde{v}, \tilde{\theta}) &\in \mathcal{M}(\Omega^T), \quad \theta_0 \in W_\varrho^{2-2/\varrho}(\Omega). \end{aligned}$$

Then there exists a unique solution to problem (3.2) such that

$$\theta \in W_\varrho^{2,1}(\Omega^T) \subset L_\infty(0, T; W_\eta^1(\Omega))$$

and

$$\|\theta\|_{L_\infty(0,T;W_\eta^1(\Omega))} \leq c\|\theta\|_{W_\varrho^{2,1}(\Omega^T)} \leq c(\lambda\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2 + \|\theta_0\|_{W_\varrho^{2-2/\varrho}(\Omega)}).$$

Proof. We have

$$\begin{aligned} \|\tilde{v} \cdot \nabla \tilde{\theta}\|_{L_\varrho(\Omega^T)} &\leq \|\tilde{v}\|_{L_\infty(\Omega^T)} \|\nabla \tilde{\theta}\|_{L_\eta(\Omega^T)} \\ &\leq c\|\tilde{v}\|_{L_\infty(0,T;W_\eta^1(\Omega))} \|\tilde{\theta}\|_{L_\infty(0,T;W_\eta^1(\Omega))} \\ &\leq c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2. \end{aligned}$$

Then similarly as in Theorem 9.1 from [2, Ch. 4, Sect. 9] (see also [6, Theorem 17]) we prove the lemma. \square

Lemma 3.4. *Let*

$$\begin{aligned} (\tilde{v}, \tilde{\theta}) &\in \mathcal{M}(\Omega^T), \quad 3 < \xi < \infty, \quad 3 < \eta < \infty \\ f &\in L_\sigma(\Omega^T), \quad g \in L_\sigma(\Omega^T), \quad 1 < \sigma < \infty \quad (\text{where } g = f_{,x_3}) \\ \sigma &< \eta, \quad S \in C^2, \quad \sigma < \xi, \quad \frac{5}{\sigma} - \frac{3}{\xi} < 1. \end{aligned}$$

Let v, p be a unique solution to problem (3.1). Let $h = v_{,x_3}$, $q = p_{,x_3}$. Assume $h(0) \in W_\sigma^{2-2/\sigma}(\Omega)$. Then

$$h \in W_\sigma^{2,1}(\Omega^T) \subset L_\infty(0, T; W_\xi^1(\Omega))$$

and

$$\begin{aligned} \|h\|_{L_\infty(0,T;W_\xi^1(\Omega))} &\leq c\|h\|_{W_\sigma^{2,1}(\Omega^T)} \leq c(\lambda\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega)}^2 \\ &\quad + \lambda a(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}\|f\|_{L_\sigma(\Omega^T)} \\ &\quad + \lambda a(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|g\|_{L_\sigma(\Omega^T)} + \|h(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}). \end{aligned}$$

Proof. The function h is a solution of the following problem

$$\begin{aligned} h_{,t} - \operatorname{div} \mathbf{T}(h, q) &= \lambda[-\tilde{v} \cdot \nabla \tilde{h} - \tilde{h} \cdot \nabla \tilde{v} + \alpha_\theta(\tilde{\theta})\tilde{v}f + \alpha(\tilde{\theta})g] && \text{in } \Omega^T, \\ \operatorname{div} h &= 0 && \text{in } \Omega^T, \\ \bar{n} \cdot h &= 0, \quad \bar{n} \cdot \mathbf{D}(h) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 && \text{on } S_1^T, \\ h_i &= 0, \quad i = 1, 2, \quad h_{3,x_3} = 0 && \text{on } S_2^T, \\ h|_{t=0} &= h(0) && \text{in } \Omega, \end{aligned}$$

where $\tilde{h} = \tilde{v}_{,x_3}$, $\tilde{\vartheta} = \tilde{\theta}_{,x_3}$.

We have

$$\begin{aligned} \|\tilde{v} \cdot \nabla \tilde{h}\|_{L_\sigma(\Omega^T)} &\leq c\|\tilde{v}\|_{L_\infty(\Omega^T)} \|\nabla \tilde{h}\|_{L_\xi(\Omega^T)} \\ &\leq c\|\tilde{v}\|_{L_\infty(0,T;W_\eta^1(\Omega))} \|\tilde{h}\|_{L_\infty(0,T;W_\xi^1(\Omega))} \\ &\leq c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2. \end{aligned}$$

and

$$\begin{aligned}
\|\tilde{h} \cdot \nabla \tilde{v}\|_{L_\sigma(\Omega^T)} &\leq c\|\tilde{h}\|_{L_\infty(\Omega^T)}\|\nabla \tilde{v}\|_{L_\eta(\Omega^T)} \\
&\leq c\|\tilde{h}\|_{L_\infty(0,T;W_\xi^1(\Omega))}\|\tilde{v}\|_{L_\infty(0,T;W_\eta^1(\Omega))} \\
&\leq c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2.
\end{aligned}$$

Then

$$\begin{aligned}
\|\alpha_\theta(\tilde{\theta})\tilde{\vartheta}f\|_{L_\sigma(\Omega^T)} &\leq ca(c\|\tilde{\theta}\|_{L_\infty(0,T;W_\eta^1(\Omega))})\|\tilde{\vartheta}\|_{L_\infty(0,T;W_\xi^1(\Omega))}\|f\|_{L_\sigma(\Omega^T)} \\
&\leq ca(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega)}\|f\|_{L_\sigma(\Omega^T)}.
\end{aligned}$$

We have

$$\begin{aligned}
\|\alpha(\tilde{\theta})g\|_{L_\sigma(\Omega^T)} &\leq a(c\|\tilde{\theta}\|_{L_\infty(0,T;W_\eta^1(\Omega))})\|g\|_{L_\sigma(\Omega^T)} \\
&\leq c(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|g\|_{L_\sigma(\Omega^T)}.
\end{aligned}$$

By Theorem 2.1 the proof is completed. \square

Lemma 3.5. Assume that $3 < \eta < \infty$, $1 < \sigma < \infty$, $\sigma < \eta$, $\frac{5}{\sigma} - \frac{3}{\xi} < 1$, $3 < \xi < \infty$, $\sigma < \xi$, $(\tilde{v}, \tilde{\theta}) \in \mathcal{M}(\Omega^T)$. Let θ be a unique solution to problem (3.2). Let $\vartheta = \theta_{,x_3}$. Assume that $\vartheta(0) \in W_\sigma^{2-2/\sigma}(\Omega)$. Then

$$\vartheta \in W_\sigma^{2,1}(\Omega^T) \subset L_\infty(0, T; W_\xi^1(\Omega))$$

and

$$\|\vartheta\|_{L_\infty(0,T;W_\xi^1(\Omega))} \leq c\|\vartheta\|_{W_\sigma^{2,1}(\Omega^T)} \leq c(\lambda\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2 + \|\vartheta(0)\|_{W_\sigma^{2-2/\sigma}(\Omega)}).$$

Proof. The function ϑ is solution of the following problem

$$\begin{aligned}
\vartheta_{,t} - \kappa \Delta \vartheta &= -\lambda[\tilde{h} \cdot \nabla \tilde{\theta} + \tilde{v} \cdot \nabla \tilde{\vartheta}] && \text{in } \Omega^T, \\
\bar{n} \cdot \nabla \vartheta &= 0 && \text{on } S_1^T, \\
\vartheta &= 0 && \text{on } S_2^T, \\
\vartheta|_{t=0} &= \vartheta(0) && \text{in } \Omega,
\end{aligned}$$

where $\tilde{\vartheta} = \tilde{\theta}_{,x_3}$. We have

$$\begin{aligned}
\|\tilde{h} \cdot \nabla \tilde{\theta}\|_{L_\sigma(\Omega^T)} &\leq \|\tilde{h}\|_{L_\infty(\Omega^T)}\|\nabla \tilde{\theta}\|_{L_\eta(\Omega^T)} \\
&\leq c\|\tilde{h}\|_{L_\infty(0,T;W_\xi^1(\Omega))}\|\tilde{\theta}\|_{L_\infty(0,T;W_\eta^1(\Omega))} \\
&\leq c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{v} \cdot \nabla \tilde{\vartheta}\|_{L_\sigma(\Omega^T)} &\leq \|\tilde{v}\|_{L_\infty(\Omega^T)}\|\nabla \tilde{\vartheta}\|_{L_\xi(\Omega^T)} \\
&\leq c\|\tilde{v}\|_{L_\infty(0,T;W_\eta^1(\Omega))}\|\tilde{\vartheta}\|_{L_\infty(0,T;W_\xi^1(\Omega))} \\
&\leq c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}^2.
\end{aligned}$$

Then similarly as in Theorem 9.1 from [2, Ch. 4, Sect. 9] (see also [6, Theorem 17]) we prove the lemma. \square

From Lemmas 3.1–3.5 it follows that if $(\tilde{v}, \tilde{\theta}) \in \mathcal{M}(\Omega^T)$, then there exists a unique solution (v, θ) to problems (3.1)–(3.2) such that $(v, \theta) \in \mathcal{M}(\Omega^T)$.

To prove the existence of solutions to problem (1.1) we apply the Leray-Schauder fixed point theorem (see [4, 7, 8]). Therefore we introduce the mapping $\phi : [0, 1] \times \mathcal{M}(\Omega^T) \rightarrow \mathcal{M}(\Omega^T)$, $(\lambda, \tilde{v}, \tilde{\theta}) \rightarrow \phi(\lambda, \tilde{v}, \tilde{\theta}) = (v, \theta)$ where (v, θ) is a solution to problems (3.1)–(3.2).

For $\lambda = 0$ we have the existence of a unique solution. For $\lambda = 1$ every fixed point is a solution to problem (1.1).

Lemma 3.6. *Let the assumptions of Lemmas 3.2–3.5 be satisfied. Then the mappings $\phi(\lambda, \cdot) : \mathcal{M}(\Omega^T) \rightarrow \mathcal{M}(\Omega^T)$, $\lambda \in [0, 1]$ are completely continuous.*

Proof. By Lemmas 3.1–3.5 the mappings $\phi(\lambda, \cdot)$, $\lambda \in [0, 1]$ are compact. From this it follows that bounded sets in $\mathcal{M}(\Omega^T)$ are transformed into bounded sets in $\mathcal{M}(\Omega^T)$. Let $(\tilde{v}_i, \tilde{\theta}_i) \in \mathcal{M}(\Omega^T)$, $i = 1, 2$ be two given elements. Then (v_i, θ_i) , $i = 1, 2$ are solutions to the problems

$$\begin{aligned} (3.3) \quad & v_{it} - \operatorname{div} \mathbf{T}(v_i, p_i) = -\lambda(\tilde{v}_i \cdot \nabla \tilde{v}_i + \alpha(\tilde{\theta}_i)f), \\ & \operatorname{div} v_i = 0 \\ & \bar{n} \cdot \mathbf{D}(v_i) \cdot \bar{\tau}|_S = 0, \quad \bar{n} \cdot v_i|_S = 0, \\ & v_i|_{t=0} = v_0, \quad i = 1, 2, \end{aligned}$$

and

$$\begin{aligned} (3.4) \quad & \theta_{it} - \kappa \Delta \theta_i = -\lambda \tilde{v}_i \cdot \nabla \tilde{\theta}_i, \\ & \bar{n} \cdot \nabla \theta_i|_S = 0, \\ & \theta_i|_{t=0} = \theta_0, \quad i = 1, 2. \end{aligned}$$

To show continuity we introduce the differences

$$(3.5) \quad V = v_1 - v_2, \quad P = p_1 - p_2, \quad \mathcal{T} = \theta_1 - \theta_2$$

which are solutions to the problems

$$\begin{aligned} (3.6) \quad & V_t - \operatorname{div} \mathbf{T}(V, P) = -\lambda[\tilde{V} \cdot \nabla \tilde{v}_1 + \tilde{v}_2 \cdot \nabla \tilde{V} + (\alpha(\tilde{\theta}_1) - \alpha(\tilde{\theta}_2))f] \\ & \operatorname{div} V = 0 \\ & V \cdot \bar{n}|_S = 0 \quad \bar{n} \cdot \mathbf{D}(V) \cdot \bar{\tau}|_S = 0, \\ & V|_{t=0} = 0 \end{aligned}$$

and

$$\begin{aligned}
(3.7) \quad & \mathcal{T}_t - \kappa \Delta \mathcal{T} = -\lambda[\tilde{V} \cdot \nabla \tilde{\theta}_1 + \tilde{v}_2 \cdot \nabla \tilde{\mathcal{T}}] \\
& \bar{n} \cdot \nabla \mathcal{T}|_S = 0, \\
& \mathcal{T}|_{t=0} = 0,
\end{aligned}$$

where $\tilde{V} = \tilde{v}_1 - \tilde{v}_2$, $\tilde{\mathcal{T}} = \tilde{\theta}_1 - \tilde{\theta}_2$.

In view of [3] and [7, 8] we have

$$\begin{aligned}
(3.8) \quad & \|V\|_{W_\varrho^{2,1}(\Omega^T)} + \|\mathcal{T}\|_{W_\varrho^{2,1}(\Omega^T)} \leq c[\|\tilde{V}\|_{L_\infty(\Omega^T)} \|\nabla \tilde{v}_1\|_{L_\varrho(\Omega^T)} \\
& + \|\tilde{v}_2\|_{L_\infty(\Omega^T)} \|\nabla \tilde{V}\|_{L_\varrho(\Omega^T)} + ca(\max\{\|\tilde{\theta}_1\|_{L_\infty(\Omega^T)}, \|\tilde{\theta}_2\|_{L_\infty(\Omega^T)}\}) \cdot \\
& \cdot \|\tilde{\mathcal{T}}\|_{L_\infty(\Omega^T)} \|f\|_{L_\varrho(\Omega^T)} + \|\tilde{v}_2\|_{L_\infty(\Omega^T)} \|\nabla \tilde{\mathcal{T}}\|_{L_\varrho(\Omega^T)} \\
& + \|\tilde{V}\|_{L_\infty(\Omega^T)} \|\nabla \tilde{\theta}_1\|_{L_\varrho(\Omega^T)}] \leq c(\|\tilde{V}\|_{\mathcal{M}(\Omega^T)} + \|\tilde{\mathcal{T}}\|_{\mathcal{M}(\Omega^T)}).
\end{aligned}$$

Let $h_i = v_{i,x_3}$, $q_i = p_{i,x_3}$, $\vartheta_i = \theta_{i,x_3}$, $\tilde{h}_i = \tilde{v}_{i,x_3}$, $\tilde{\vartheta}_i = \tilde{\theta}_{i,x_3}$.

The functions h_i, ϑ_i , $i = 1, 2$ are solutions to the following problems

$$\begin{aligned}
& h_{i,t} - \operatorname{div} \mathbf{T}(h_i, q_i) = -\lambda[\tilde{h}_i \cdot \nabla \tilde{v}_i + \tilde{v}_i \cdot \nabla \tilde{h}_i + \alpha_\theta(\tilde{\theta}_i) \tilde{\vartheta}_i f + \alpha(\tilde{\theta}_i) g] & \text{in } \Omega^T, \\
& \operatorname{div} h_i = 0 & \text{in } \Omega^T, \\
& \bar{n} \cdot h_i = 0, \quad \bar{n} \cdot \mathbf{D}(h) \cdot \bar{\tau}_\alpha, \quad \alpha = 1, 2, \quad i = 1, 2 & \text{on } S_1^T, \\
& h_{ij} = 0, \quad i = 1, 2, \quad j = 1, 2 & \text{on } S_2^T, \\
& h_{i3,x_3} = 0, \quad i = 1, 2 & \text{on } S_2^T, \\
& h_i|_{t=0} = h(0) & \text{in } \Omega
\end{aligned}$$

and

$$\begin{aligned}
& \vartheta_{i,t} - \kappa \Delta \vartheta_i = -\lambda[\tilde{h}_i \cdot \nabla \tilde{\theta}_i + \tilde{v}_i \cdot \nabla \tilde{\vartheta}_i] & \text{in } \Omega^T, \\
& \bar{n} \cdot \nabla \vartheta_i = 0 & \text{on } S_1^T, \\
& \vartheta_i = 0 & \text{on } S_2^T, \\
& \vartheta_i|_{t=0} = \vartheta(0) & \text{in } \Omega.
\end{aligned}$$

We introduce the differences

$$H = h_1 - h_2, \quad Q = q_1 - q_2, \quad R = \vartheta_1 - \vartheta_2$$

which are solutions to the problems

$$\begin{aligned}
& H_{,t} - \operatorname{div} \mathbf{T}(H, Q) = -\lambda[\tilde{H} \cdot \nabla \tilde{v}_1 + \tilde{h}_2 \cdot \nabla \tilde{V} + \tilde{V} \cdot \nabla \tilde{h}_1 + \tilde{v}_2 \cdot \nabla \tilde{H} \\
& + (\alpha_\theta(\tilde{\theta}_1) - \alpha_\theta(\tilde{\theta}_2)) \tilde{\vartheta}_1 f + \alpha_\theta(\tilde{\theta}_2) \tilde{R} f \\
& + (\alpha(\tilde{\theta}_1) - \alpha(\tilde{\theta}_2)) g & \text{in } \Omega^T, \\
& \operatorname{div} H = 0 & \text{in } \Omega^T, \\
& \bar{n} \cdot H = 0, \quad \bar{n} \cdot \mathbf{D}(H) \cdot \bar{\tau}_\alpha = 0, \quad \alpha = 1, 2 & \text{on } S_1^T, \\
& H_j = 0, \quad j = 1, 2, \quad H_{3,x_3} = 0 & \text{on } S_2^T, \\
& H|_{t=0} = 0 & \text{in } \Omega.
\end{aligned}$$

and

$$\begin{aligned} R_{,t} - \kappa \Delta R &= -\lambda[\tilde{H} \cdot \nabla \tilde{\theta}_1 + \tilde{h}_2 \cdot \nabla \tilde{\mathcal{T}} + \tilde{V} \cdot \nabla \tilde{\vartheta}_1 + \tilde{v}_2 \cdot \nabla \tilde{R}] & \text{in } \Omega^T, \\ \bar{n} \cdot \nabla R &= 0 & \text{on } S_1^T, \\ R &= 0 & \text{on } S_2^T, \\ R|_{t=0} &= 0 & \text{in } \Omega, \end{aligned}$$

where $\tilde{H} = \tilde{h}_1 - \tilde{h}_2$, $\tilde{R} = \tilde{\vartheta}_1 - \tilde{\vartheta}_2$. In view of [3] and [7, 8] we have

$$\begin{aligned} \|H\|_{W_{\sigma}^{2,1}(\Omega^T)} + \|R\|_{W_{\sigma}^{2,1}(\Omega^T)} &\leq c[\|\tilde{H}\|_{L_{\infty}(\Omega^T)} \|\nabla \tilde{v}_1\|_{L_{\eta}(\Omega^T)} \\ &+ \|\tilde{h}_2\|_{L_{\infty}(\Omega^T)} \|\nabla \tilde{V}\|_{L_{\eta}(\Omega^T)} + \|\tilde{V}\|_{L_{\infty}(\Omega^T)} \|\nabla \tilde{h}_1\|_{L_{\xi}(\Omega^T)} \\ &+ \|\tilde{v}_2\|_{L_{\infty}(\Omega^T)} \|\nabla \tilde{H}\|_{L_{\xi}(\Omega^T)} + c(\max\{\|\tilde{\theta}_1\|_{L_{\infty}(\Omega^T)}, \|\tilde{\theta}_2\|_{L_{\infty}(\Omega^T)}\}) \\ &\cdot \|\mathcal{T}\|_{L_{\infty}(\Omega^T)} \|\tilde{\vartheta}_1\|_{L_{\infty}(\Omega^T)} \|f\|_{L_{\sigma}(\Omega^T)} \\ &+ c(\|\tilde{\theta}_2\|_{L_{\infty}(\Omega^T)} \|\tilde{R}\|_{L_{\infty}(\Omega^T)} \|f\|_{L_{\sigma}(\Omega^T)} \\ &+ a(\max\{\|\tilde{\theta}_1\|_{L_{\infty}(\Omega^T)}, \|\tilde{\theta}_2\|_{L_{\infty}(\Omega^T)}\}) \|\tilde{\mathcal{T}}\|_{L_{\infty}(\Omega^T)} \|g\|_{L_{\sigma}(\Omega^T)} \\ &+ \|\tilde{H}\|_{L_{\infty}(\Omega^T)} \|\nabla \tilde{\theta}_1\|_{L_{\eta}(\Omega^T)} + \|\tilde{h}_2\|_{L_{\infty}(\Omega^T)} \|\nabla \tilde{\mathcal{T}}\|_{L_{\eta}(\Omega^T)} \\ &+ \|\tilde{V}\|_{L_{\infty}(\Omega^T)} \|\nabla \tilde{\vartheta}_1\|_{L_{\xi}(\Omega^T)} + \|\tilde{v}_2\|_{L_{\infty}(\Omega^T)} \|\nabla \tilde{R}\|_{L_{\xi}(\Omega^T)}] \leq c(\|\tilde{V}, \tilde{\mathcal{T}}\|_{\mathcal{M}(\Omega^T)}) \end{aligned}$$

and from (3.8) and Lemma 3.1 we obtain

$$\|(V, \mathcal{T})\|_{\mathcal{M}(\Omega^T)} \leq c\|(\tilde{V}, \tilde{\mathcal{T}})\|_{\mathcal{M}(\Omega^T)}.$$

Hence continuity of ϕ follows. This concludes the proof. \square

Lemma 3.7. *Let assumptions of Lemmas 3.2–3.5 be satisfied. Then for every bounded subset \mathcal{M}_0 of $\mathcal{M}(\Omega^T)$, the family of maps*

$$\phi(\cdot, \tilde{v}, \tilde{\theta}) : [0, 1] \rightarrow \mathcal{M}(\Omega^T), \quad (\tilde{v}, \tilde{\theta}) \in \mathcal{M}_0$$

is uniformly equicontinuous.

Proof. Let $(\tilde{v}, \tilde{\theta}) \in \mathcal{M}_0$, $\lambda_i \in [0, 1]$, $i = 1, 2$, $\lambda_1 \geq \lambda_2$ and v_i, θ_i are solutions to the problems

$$\begin{aligned} v_{it} - \operatorname{div} \mathbf{T}(v_i, p_i) &= -\lambda_i(\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta})f), \\ \operatorname{div} v_i &= 0, \\ \bar{n} \cdot \mathbf{D}(v_i) \cdot \bar{\tau}|_S &= 0, \quad \bar{n} \cdot v_i|_S = 0, \\ v_i|_{t=0} &= v_0, \quad i = 1, 2 \end{aligned}$$

and

$$\begin{aligned} \theta_{it} - \kappa \Delta \theta_i &= -\lambda_i \tilde{v} \cdot \nabla \tilde{\theta}, \\ \bar{n} \cdot \nabla \theta_i|_S &= 0, \\ \theta_i|_{t=0} &= \theta_0, \quad i = 1, 2. \end{aligned}$$

To show uniform equicontinuity we introduce the differences

$$V = v_1 - v_2, \quad P = p_1 - p_2, \quad \mathcal{T} = \theta_1 - \theta_2$$

which are solutions to the problems

$$\begin{aligned} V_t - \operatorname{div} \mathbf{T}(V, P) &= -(\lambda_1 - \lambda_2)(\tilde{v} \cdot \nabla \tilde{v} + \alpha(\tilde{\theta})f), \\ \operatorname{div} V &= 0, \\ \bar{n} \cdot \mathbf{D}(V) \cdot \bar{\tau}|_S &= 0, \quad \bar{n} \cdot V|_S = 0, \\ V|_{t=0} &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}_t - \kappa \Delta \mathcal{T} &= -(\lambda_1 - \lambda_2)\tilde{v} \cdot \nabla \tilde{\theta}, \\ \bar{n} \cdot \nabla \mathcal{T}|_S &= 0, \\ \mathcal{T}|_{t=0} &= 0. \end{aligned}$$

In view of Lemmas 3.2–3.3

$$\begin{aligned} (3.9) \quad & \|V\|_{L_\infty(0,T;W_\eta^1(\Omega))} + \|\mathcal{T}\|_{L_\infty(0,T;W_\eta^1(\Omega))} \leq c((\lambda_1 - \lambda_2)\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega)}^2 \\ & + (\lambda_1 - \lambda_2)a(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|f\|_{L_\varphi(\Omega^T)}). \end{aligned}$$

Let $h_i = v_{i,x_3}$, $\vartheta_i = \theta_{i,x_3}$.

We introduce the differences

$$H = h_1 - h_2, \quad R = \vartheta_1 - \vartheta_2$$

which satisfy the following conditions

$$H = V_{,x_3}, \quad R = \mathcal{T}_{,x_3}.$$

In view of Lemmas 3.4 and 3.5

$$\begin{aligned} (3.10) \quad & \|H\|_{L_\infty(0,T;W_\xi^1(\Omega))} + \|R\|_{L_\infty(0,T;W_\xi^1(\Omega))} \leq c((\lambda_1 - \lambda_2)\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)} \\ & + (\lambda_1 - \lambda_2)a(c\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)})\|(\tilde{v}, \tilde{\theta})\|_{\mathcal{M}(\Omega^T)}\|f\|_{L_\sigma(\Omega^T)} \\ & + (\lambda_1 - \lambda_2)a(c\|(\tilde{v}, \theta)\|_{\mathcal{M}(\Omega^T)})\|g\|_{L_\sigma(\Omega^T)}). \end{aligned}$$

From (3.9) and (3.10) the uniform equicontinuity of $\phi(\cdot, \tilde{v}, \tilde{\theta})$ follows.

Proof of Theorem 1.1.

In view of the above considerations and [5, Main Theorem] the assumptions of the Leray-Schauder fixed point theorem are satisfied. Hence Theorem 1.1 is proved.

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